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Weak tensor products of complete atomistic lattices

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Abstract. Given two complete atomistic lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, we define a set $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ of complete atomistic lattices by means of three axioms (natural regarding the description of *separated* quantum compound systems), or in terms of a universal property with respect to a given class of bimorphisms. We call the elements of $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ weak tensor products of $\mathcal{L}_1$ and $\mathcal{L}_2$. We prove that $\mathsf{S}(\mathcal{L}_1, \mathcal{L}_2)$ is a complete lattice. We compare the bottom element $\mathcal{L}_1 \owedge \mathcal{L}_2$ with the *separated product* of Aerts and with the *box product* of Grätzer and Wehrung. Similarly, we compare the top element $\mathcal{L}_1 \ovee \mathcal{L}_2$ with the tensor products of Fraser, Chu and Shmuely. With some additional hypotheses on $\mathcal{L}_1$ and $\mathcal{L}_2$ (true for instance if $\mathcal{L}_1$ and $\mathcal{L}_2$ are moreover irreducible, orthocomplemented and with the covering property), we characterize the automorphisms of weak tensor products in terms of those of $\mathcal{L}_1$ and $\mathcal{L}_2$.

1. Introduction

In quantum logic, one associates with a physical system S a couple $(\Sigma_S, \mathcal{L}_S \subseteq 2^{\Sigma_S})$, where Σ_S represents the set of all possible states of S and $\mathcal{L}_S$, the set of experimental propositions concerning S: A proposition represented by some $a \in \mathcal{L}_S$ is true with probability 1 if and only if the state of S lies in a [3]. It is usually assumed that $\mathcal{L}_S$ is a *simple closure space* on Σ_S — in other words, that $\mathcal{L}_S$ is closed under arbitrary intersections, and contains $\emptyset$, Σ_S, and all singletons of Σ_S. In particular, then $\mathcal{L}_S$ is a complete atomistic lattice. Note that any complete atomistic lattice is isomorphic to a simple closure space on its set of atoms.

If S is a compound system consisting of two *separated* quantum systems S_1 and S_2 (for instance two electrons prepared in two different rooms of the lab), then we have [1, 8]

$$\Sigma_S = \Sigma_{S_1} \times \Sigma_{S_2}. \tag{P1}$$

In particular, simultaneous experiments on both systems can be performed, and any experiment done on one system does not alter the state of the other system. Therefore, if P_1 is a proposition concerning S_1 represented by some $a_1 \in \mathcal{L}_{S_1}$ and P_2

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is a proposition concerning S_2 represented by some $a_2 \in \mathcal{L}_{S_2}$, then the proposition P_1 OR P_2 concerning the compound system is true with probability 1 if and only if the state p_1 of S_1 lies in a_1 or the state p_2 of S_2 lies in a_2. In other words [1, 8]

(P2) $\quad a_1 \times \Sigma_{S_2} \cup \Sigma_{S_1} \times a_2 \in \mathcal{L}_S$, for all $a_1 \in \mathcal{L}_{S_1}$ and $a_2 \in \mathcal{L}_{S_2}$.

Moreover, in addition to Axioms P1 and P2 above, we postulate:

(P3) $\quad$ For all $p_i \in \Sigma_{S_i}$ and $A_i \subseteq \Sigma_{S_i}$, $[p_1 \times A_2 \in \mathcal{L}_S \Rightarrow A_2 \in \mathcal{L}_2]$ and $[A_1 \times p_2 \in \mathcal{L}_S \Rightarrow A_1 \in \mathcal{L}_1]$.

See [8] for detailed physical justifications of Axioms P1–P3. We define $\mathsf{S}(\mathcal{L}_{S_1}, \mathcal{L}_{S_2})$ to be the set of all simple closure spaces on $\Sigma_{S_1} \times \Sigma_{S_2}$ for which Axioms P2 and P3 hold.

The rest of the paper is organized as follows. In Section 2, we fix some basic terminology and notation. We define the set $\mathsf{S} \equiv \mathsf{S}(\mathcal{L}_1, \dots, \mathcal{L}_n)$ of n-fold weak tensor products. We prove that S is a complete lattice, the bottom and top elements of which are denoted by $\textcircled{\scriptsize$\wedge$}_i \mathcal{L}_i$ and $\textcircled{\scriptsize$\vee$}_i \mathcal{L}_i$ respectively. In Section 3 we compare $\textcircled{\scriptsize$\wedge$}$ and $\textcircled{\scriptsize$\vee$}$ with several other tensor products of lattices. An equivalent definition of S, in terms of a universal property with respect to a given class of multimorphisms (arbitrary joins are preserved), is given in Section 4. In Section 5 we prove some basic relations between the central elements of $\mathcal{L}_i$'s and those of $\textcircled{\scriptsize$\vee$}_\alpha \mathcal{L}_\alpha$ and $\textcircled{\scriptsize$\wedge$}_\alpha \mathcal{L}_\alpha$.

Let $\mathcal{L} \in \mathsf{S}$ and let $u \colon \mathcal{L} \to \mathcal{L}$ preserve arbitrary joins and send atoms to atoms. In Section 6 we prove, under some hypotheses on the image of u and on each $\mathcal{L}_i$ (true for instance if $\mathcal{L}_i$ is moreover irreducible, orthocomplemented and with the covering property), that there exists a permutation f of $\{1, 2, \dots, n\}$, and join-preserving maps $v_i \colon \mathcal{L}_i \to \mathcal{L}_{f(i)}$ sending atoms to atoms such that on atoms, $u = f \circ (v_1 \times \cdots \times v_n)$. A time evolution can be modelled by a map preserving arbitrary joins and sending atoms to atoms [4]. Hence, in the physical interpretation, this result shows that separated quantum systems remain separated only if they do not interact.

2. Main definitions

In this section we give our main definitions. We start with some background material and basic notations used in the sequel.

Definition 2.1. Let Σ be a non-empty set and ω a set of subsets of Σ. We write $\bigcap \omega$ (respectively $\bigcup \omega$) instead of $\bigcap_{a \in \omega} a$ (respectively $\bigcup_{a \in \omega} a$). By a *simple closure space* $\mathcal{L}$ on Σ we mean a set of subsets of Σ, ordered by set-inclusion, closed under arbitrary set-intersections (*i.e.*, for all $\omega \subseteq \mathcal{L}$, $\bigcap \omega \in \mathcal{L}$), and containing Σ, $\emptyset$, and all singletons. We denote the bottom ($\emptyset$) and top (Σ) elements by 0 and 1 respectively. For $p \in \Sigma$, we identify p with $\{p\} \in \mathcal{L}$. Hence $p \cup q$ stands for $\{p, q\}$.

We denote the category of simple closure spaces with maps preserving arbitrary joins (hence 0) by **Cl**, and the sub-category of simple closure spaces on a particular (nonempty) set Σ, by $\mathbf{Cl}(\Sigma)$. Let 2 denote the simple closure space isomorphic to the two-element lattice.

Let $\mathcal{L}$, $\mathcal{L}_1 \in \mathbf{Cl}(\Sigma)$ and $u\colon \mathcal{L} \to \mathcal{L}_1$ a map sending atoms to atoms. We write $\mathsf{Aut}(\mathcal{L})$ for the group of automorphisms of $\mathcal{L}$, and we also call u the mapping from Σ to Σ induced by u.

Remark 2.2. Let $\mathcal{L}$ be a simple closure space on a (nonempty) set Σ. Then $\mathcal{L}$ is a complete atomistic lattice, the atoms of which correspond to the points (*i.e.*, singletons) of Σ. Note that if $A \subseteq \Sigma$, then $\bigvee_{\mathcal{L}}(A) = \bigcap\{b \in \mathcal{L} \ ; \ A \subseteq b\}$.

Conversely, let $\mathcal{L}$ be a complete atomistic lattice. Let Σ denote the set of atoms of $\mathcal{L}$, and, for each $a \in \mathcal{L}$, let $\Sigma[a]$ denote the set of atoms under a. Then $\{\Sigma[a] \ ; \ a \in \mathcal{L}\}$ is a simple closure space on Σ, isomorphic to $\mathcal{L}$.

Finally, let $\mathcal{L}$ be a simple closure space and let $u \in \mathsf{Aut}(\mathcal{L})$. Then note that for all $a \in \mathcal{L}$, $u(a) = \{u(p) \ ; \ p \in a\}$.

Definition 2.3. Let $\{\Sigma_\alpha\}_{\alpha\in\Omega}$ be a family of nonempty sets, $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, $p \in \mathbf{\Sigma}$, $R \subseteq \mathbf{\Sigma}$, $A \in \prod_\alpha 2^{\Sigma_\alpha}$, $\beta \in \Omega$, and $B \subseteq \Sigma_\beta$. We shall make use of the following notations:

(1) We denote by $\pi_\beta\colon \mathbf{\Sigma} \to \Sigma_\beta$ the $\beta-$th coordinate map, *i.e.*, $\pi_\beta(p) = p_\beta$.
(2) We denote by $p[-,\beta]\colon \Sigma_\beta \to \mathbf{\Sigma}$ the map that sends $q \in \Sigma_\beta$ to the element of $\mathbf{\Sigma}$ obtained by replacing p's $\beta-$th entry by q.
(3) We define $R_\beta[p] = \pi_\beta(p[\Sigma_\beta,\beta] \cap R)$. Note that $R_\beta[p] = \{q \in \Sigma_\beta \ ; \ p[q,\beta] \in R\}$.
(4) We define $A[B,\beta] \in \prod_\alpha 2^{\Sigma_\alpha}$ as $A[B,\beta]_\beta = B$ and $A[B,\beta]_\alpha = A_\alpha$ for $\alpha \neq \beta$.
(5) We write $\overline{A} := \prod_\alpha A_\alpha$ and $\overline{A}[B,\beta] := \overline{A[B,\beta]}$.

We omit the β in $p[-,\beta]$ when no confusion can occur. For instance, we write $p[\Sigma_\beta]$ instead of $p[\Sigma_\beta,\beta]$.

Remark 2.4. $p[R_\beta[p]] = p[\Sigma_\beta] \cap R$.

Definition 2.5. Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α. We denote by $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ the set all $\mathcal{L} \in \mathbf{Cl}(\mathbf{\Sigma})$ such that

(P1) $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$,
(P2) $\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha) \in \mathcal{L}$, for all $a \in \prod_\alpha \mathcal{L}_\alpha$,
(P3) for all $p \in \mathbf{\Sigma}$, $\beta \in \Omega$, and $B \subseteq \Sigma_\beta$, $[p[B,\beta] \in \mathcal{L} \Rightarrow B \in \mathcal{L}_\beta]$.

Let $T = \prod_\alpha T_\alpha$ with $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$. We denote by $\mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$ the set of all $\mathcal{L} \in \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ such that

(P4) for all $v \in T$, there is $u \in \mathsf{Aut}(\mathcal{L})$ such that $u(p)_\alpha = v_\alpha(p_\alpha)$ for all $p \in \mathbf{\Sigma}$ and all $\alpha \in \Omega$.

We call elements of $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ *weak tensor products* of the family $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$.

Remark 2.6. The u in Axiom P4 is necessarily unique. Note also that for all $T = \prod_\alpha T_\alpha$, with $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$, $\mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega) \subseteq \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$. The name "weak tensor product" is justified by the fact that $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ and $\mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$ can be defined in terms of a universal property with respect to a given class of multimorphisms of **Cl** (see Section 4).

Lemma 2.7. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α, $\beta \in \Omega$, and $\mathcal{L} \in \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$.*

(1) *For any $a \in \prod_\alpha \mathcal{L}_\alpha$, $\overline{a} \in \mathcal{L}$.*
(2) *For any $b \in \mathcal{L}_\beta$ and every $p \in \prod_\alpha \Sigma_\alpha$, $p[b, \beta] \in \mathcal{L}$.*
(3) *For any $B \subseteq \mathcal{L}_\beta$ and every $a \in \prod_\alpha \mathcal{L}_\alpha$, $\overline{a}[\bigvee B, \beta] = \bigvee_{b\in B} \overline{a}[b, \beta]$.*

Proof. (1) Let $\beta \in \Omega$. Define $\widehat{a}^\beta \in \prod_\alpha \mathcal{L}_\alpha$ by setting $\widehat{a}^\beta_\alpha = a_\beta$ if $\alpha = \beta$, and $\emptyset$ otherwise. Note that $\pi_\beta^{-1}(a_\beta) = \bigcup_\alpha \pi_\alpha^{-1}(\widehat{a}^\beta_\alpha)$. Now, by Axiom P2, $\bigcup_\alpha \pi_\alpha^{-1}(\widehat{a}^\beta_\alpha) \in \mathcal{L}$, therefore $\pi_\beta^{-1}(a_\beta) \in \mathcal{L}$. Finally $\overline{a} = \bigcap_\alpha \pi_\alpha^{-1}(a_\alpha)$, hence $\overline{a} \in \mathcal{L}$.

(2) Define $a \in \prod_\alpha \mathcal{L}_\alpha$ as $a_\alpha = p_\alpha$ if $\alpha \neq \beta$ and $a_\beta = b$. Then $p[b, \beta] = \prod_\alpha a_\alpha$, hence from the first part $p[b, \beta] \in \mathcal{L}$.

(3) By the first part, $\overline{a}[\bigvee B, \beta] \in \mathcal{L}$. Moreover, $\overline{a}[b, \beta] \subseteq \overline{a}[\bigvee B, \beta]$ for all $b \in B$, hence $\bigvee_{b\in B} \overline{a}[b, \beta] \subseteq \overline{a}[\bigvee B, \beta]$. As a consequence, there is $X \subseteq \Sigma_\beta$ such that $\bigvee_{b\in B} \overline{a}[b, \beta] = \overline{a}[X, \beta]$ and $X \subseteq \bigvee B$. Moreover, $\overline{a}[X, \beta] \in \mathcal{L}$, therefore $p[X, \beta] \in \mathcal{L}$ for any $p \in \overline{a}$, whence, by Axiom P3, it follows that $X \in \mathcal{L}_\beta$. Finally, $\bigcup B \subseteq X$, therefore $\bigvee B \subseteq X$. As a consequence, $X = \bigvee B$. □

Lemma 2.8. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α, $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, and $\mathcal{L}_0$, $\mathcal{L}$ and $\mathcal{L}_1$ be simple closures spaces on $\mathbf{\Sigma}$. Suppose that $\mathcal{L}_0 \subseteq \mathcal{L} \subseteq \mathcal{L}_1$.*

(1) *If Axiom* P2 *holds in $\mathcal{L}_0$, then it holds also in $\mathcal{L}$.*
(2) *If Axiom* P3 *holds in $\mathcal{L}_1$, then it holds also in $\mathcal{L}$.*

Proof. Direct from Definition 2.5. □

Definition 2.9. Let $\{\Sigma_\alpha\}_{\alpha\in\Omega}$ be a family of nonempty sets and $\{\mathcal{L}_\alpha \subseteq 2^{\Sigma_\alpha}\}_{\alpha\in\Omega}$. Let $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$. Then

$$\textcircled{\scriptsize$\wedge$}\{\mathcal{L}_\alpha \,;\, \alpha \in \Omega\} := \Big\{ \bigcap \omega \,;\, \omega \subseteq \big\{ \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha) \,;\, a \in \prod_\alpha \mathcal{L}_\alpha \big\} \Big\},$$

$$\textcircled{\scriptsize$\vee$}\{\mathcal{L}_\alpha \,;\, \alpha \in \Omega\} := \big\{ R \subseteq \mathbf{\Sigma} \,;\, R_\beta[p] \in \mathcal{L}_\beta, \forall p \in \mathbf{\Sigma},\, \beta \in \Omega \big\},$$

ordered by set-inclusion.

Lemma 2.10. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α with $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, and let $\mathcal{L}_0$ and $\mathcal{L}_1$ be simple closures spaces on $\mathbf{\Sigma}$.*

(1) *If Axiom* P2 *holds in $\mathcal{L}_0$, then $\textcircled{\scriptsize$\wedge$}_\alpha \mathcal{L}_\alpha \subseteq \mathcal{L}_0$.*
(2) *If Axiom* P3 *holds in $\mathcal{L}_1$ and if $p[\Sigma_\beta, \beta] \in \mathcal{L}_1$, for all $p \in \mathbf{\Sigma}$ and all $\beta \in \Omega$, then $\mathcal{L}_1 \subseteq \textcircled{\scriptsize$\vee$}_\alpha \mathcal{L}_\alpha$.*

Proof. (1) Direct from Definitions 2.5 and 2.9.

(2) Let $R \in \mathcal{L}_1$, $p \in \mathbf{\Sigma}$, and $\beta \in \Omega$. By hypothesis, $p[\Sigma_\beta, \beta] \in \mathcal{L}_1$, hence $p[\Sigma_\beta, \beta] \cap R \in \mathcal{L}_1$. Now, $p[\Sigma_\beta, \beta] \cap R = p[R_\beta[p]]$ (see Remark 2.4). As a consequence, $p[R_\beta[p]] \in \mathcal{L}_1$ therefore, by Axiom P3, $R_\beta[p] \in \mathcal{L}_\beta$. $\square$

Proposition 2.11. *Let $\{\Sigma_\alpha\}_{\alpha\in\Omega}$ be a family of nonempty sets, and let $\{\mathcal{L}_\alpha \subseteq 2^{\Sigma_\alpha}\}_{\alpha\in\Omega}$. Let $\{\Omega_\gamma \subseteq \Omega \ ; \ \gamma \in \Gamma\}$ such that $\Omega = \coprod\{\Omega_\gamma \ ; \ \gamma \in \Gamma\}$. Then*

$$\bigcirc\!\!\!\!\wedge\{\mathcal{L}_\alpha \ ; \ \alpha \in \Omega\} = \bigcirc\!\!\!\!\wedge_{\gamma\in\Gamma}(\bigcirc\!\!\!\!\wedge_{\alpha\in\Omega_\gamma}\mathcal{L}_\alpha)\,,$$
$$\bigcirc\!\!\!\!\vee\{\mathcal{L}_\alpha \ ; \ \alpha \in \Omega\} = \bigcirc\!\!\!\!\vee_{\gamma\in\Gamma}(\bigcirc\!\!\!\!\vee_{\alpha\in\Omega_\gamma}\mathcal{L}_\alpha)\,.$$

Proof. Direct from Definition 2.9. $\square$

Theorem 2.12. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α, $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, and $T = \prod_\alpha T_\alpha$ with $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$.*

(1) *$\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$ and $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$ are simple closure spaces on $\mathbf{\Sigma}$.*
(2) *$\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$, $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha \in \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$*
(3) *$\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega) = \big\{\mathcal{L} \in \mathbf{Cl}(\mathbf{\Sigma}) \ ; \ \bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha \subseteq \mathcal{L} \subseteq \bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha\big\}$*
(4) *$\mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$ and $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$, ordered by set-inclusion, are complete lattices.*

Proof. (1) Obviously, $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$ and $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$ contain $\emptyset$ and $\mathbf{\Sigma}$, and by definition $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$ is $\cap-$closed and $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$ contains all singletons of $\mathbf{\Sigma}$. If $\omega \subseteq \bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$, then $(\bigcap\omega)_\beta[p] = \bigcap\{R_\beta[p] \ ; \ R \in \omega\}$. Moreover, for all $p \in \mathbf{\Sigma}$, $\bigcap_\alpha \pi_\alpha^{-1}(p_\alpha) = p$. As a consequence, $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$ and $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$ are simple closure spaces on $\mathbf{\Sigma}$.

(2) By definition, Axiom P2 holds in $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$ and Axiom P3 holds in $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$.

Let $a \in \prod_\alpha \mathcal{L}_\alpha$, $\beta \in \Omega$, $p \in \mathbf{\Sigma}$, and $R = \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)$. Then $R_\beta[p] = \Sigma_\beta$ if $p_\alpha \in a_\alpha$ for some $\alpha \neq \beta$, and $R_\beta[p] = a_\beta$ otherwise. As a consequence, $\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha) \in \bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$, hence Axiom P2 holds in $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$, therefore $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha \in \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$.

By Lemma 2.10 part (1), $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha \subseteq \bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$, hence by Lemma 2.8, Axiom P3 holds in $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$. As a consequence, $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha \in \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$.

Finally, let $v \in T$, $a \in \prod_\alpha \mathcal{L}_\alpha$, $p \in \mathbf{\Sigma}$, $R \subseteq \mathbf{\Sigma}$, and $\omega \subseteq 2^{\mathbf{\Sigma}}$. Then

$$v(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)) = \bigcup_\alpha \pi_\alpha^{-1}(v_\alpha(a_\alpha))\,,$$

and

$$v(R)_\beta[v(p)] = v_\beta(R_\beta[p])\,.$$

Moreover $v(\bigcap\omega) = \bigcap\{v(x) \ ; \ x \in \omega\}$. Hence, v is a bijection of $\bigcirc\!\!\!\!\wedge_\alpha\mathcal{L}_\alpha$ and of $\bigcirc\!\!\!\!\vee_\alpha\mathcal{L}_\alpha$, and v preserves arbitrary meets, hence also arbitrary joins.

(3) Follows directly from Lemmata 2.8 and 2.10.

(4) Let $\omega \subseteq \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ (respectively $\omega \subseteq \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$). Then obviously $\bigcap\omega = \{a \in \mathcal{L}\,,\, \forall\mathcal{L} \in \omega\} \in \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ (respectively $\bigcap\omega \in \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$). $\square$

Remark 2.13. Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α. For all $T = \prod_\alpha T_\alpha$ with $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$, $\mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$ is a complete meet-sublattice of $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$. Moreover, $\textcircled{\scriptsize$\wedge$}_\alpha \mathcal{L}_\alpha$ and $\textcircled{\scriptsize$\vee$}_\alpha \mathcal{L}_\alpha$ are the bottom and the top elements of $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ respectively.

Proposition 2.14. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be simple closure spaces. Then $\mathcal{L}_1 \textcircled{\scriptsize$\wedge$} \mathcal{L}_2 \cong \mathcal{L}_2 \textcircled{\scriptsize$\wedge$} \mathcal{L}_1$, $\mathcal{L}_1 \textcircled{\scriptsize$\vee$} \mathcal{L}_2 \cong \mathcal{L}_2 \textcircled{\scriptsize$\vee$} \mathcal{L}_1$, and $2 \textcircled{\scriptsize$\wedge$} \mathcal{L}_1 \cong \mathcal{L}_1$.*

Proof. Direct from Definition 2.9. □

More generally, then we have:

Proposition 2.15. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α, and $T = \prod_\alpha T_\alpha$ with $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$. Then there is an isomorphism $f\colon \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega) \to \mathcal{S}_T(2, \mathcal{L}_\alpha, \alpha \in \Omega)$ such that for all $\mathcal{L} \in \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$, $\mathcal{L} \cong f(\mathcal{L})$.*

Proof. Direct from Definition 2.5. □

Example 2.16. We now consider, for contrast, two examples, well known in many-body quantum physics, where instead of Axioms P1 and P2, we have

(p1) $\exists f\colon \prod_\alpha \Sigma_\alpha \to \boldsymbol{\Sigma}$ with f injective,
(p2) $f(\prod_\alpha \Sigma_\alpha) \cap (\bigvee f(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha))) = f(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha))$, $\forall a \in \prod_\alpha \mathcal{L}_\alpha$.

If $\mathcal{H}$ is a complex Hilbert space, then $\Sigma_{\mathcal{H}}$ denotes the set of one-dimensional subspaces of $\mathcal{H}$ and $\mathsf{P}(\mathcal{H})$ stands for the simple closure space isomorphic to the lattice of closed subspaces of $\mathcal{H}$. Moreover, we write $\mathsf{U}(\mathcal{H})$ for the group of automorphisms of $\mathsf{P}(\mathcal{H})$ induced by unitary maps on $\mathcal{H}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces, $\mathcal{L}_1 = \mathsf{P}(\mathcal{H}_1)$, $\mathcal{L}_2 = \mathsf{P}(\mathcal{H}_2)$ and $\mathcal{L} = \mathsf{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then Axioms p1 (with $\boldsymbol{\Sigma} = \Sigma_{\mathcal{H}_1\otimes\mathcal{H}_2}$), p2 and p3 (replace $p[B,\beta]$ by $f(p[B,\beta])$ in Axiom P3) hold in $\mathcal{L}$. Moreover, Axiom p4 (replace $u(p)_\alpha = v_\alpha(p_\alpha)$ by $f^{-1}(u(f(p)))_\alpha = v_\alpha(p_\alpha)$ in Axiom P4) holds for $T = \mathsf{U}(\mathcal{H}_1) \times \mathsf{U}(\mathcal{H}_2)$. Note that $\bigvee(A_1 \times A_2) = 1$, for all $A_i \subseteq \Sigma_{\mathcal{H}_i}$ with $\bigvee A_i = 1$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}$ be the Fock space (neither symmetrized, nor antisymmetrized). Let $\mathcal{L} = \mathsf{P}(\mathcal{F})$, and for each integer i, let $\mathcal{L}_i = \mathsf{P}(\mathcal{H})$. Let n be an integer and consider the family $\{\mathcal{L}_i\ ;\ 1 \leq i \leq n\}$. Then Axioms p1, p2, p3 and p4 with $T_i = \mathsf{U}(\mathcal{H})$ hold in $\mathcal{L}$ for all n. For all i, let $\Sigma_i = \Sigma_{\mathcal{H}}$. Note that in that case, for all n we have $\bigvee(\prod_{i=1}^n \Sigma_i) \neq 1$.

3. Comparison with other tensor products

In this section, we compare the bottom element $\mathcal{L}_1 \textcircled{\scriptsize$\wedge$} \mathcal{L}_2$ with the *separated product* of Aerts and with the *box product* $\mathcal{L}_1 \Box \mathcal{L}_2$ of Grätzer and Wehrung. On the other hand, we compare the top element $\mathcal{L}_1 \textcircled{\scriptsize$\vee$} \mathcal{L}_2$ with the semilattice tensor product of Fraser and with the tensor products of Chu and Shmuely.

3.1. The separated product.

Definition 3.1. A lattice $\mathcal{L}$ with 0 and 1 is *orthocomplemented* if there is a unary operation ${}^\perp\colon \mathcal{L} \to \mathcal{L}$ such that for all a, $b \in \mathcal{L}$, $(a^\perp)^\perp = a$, $a \leq b$ implies $b^\perp \leq a^\perp$, and $a \bigwedge a^\perp = 0$.

Let $\mathcal{L}$ be an orthocomplemented simple closure space on Σ. Then for p, $q \in \Sigma$, we write $p \perp q$ if $p \in q^\perp$, where $q^\perp$ stands for $\{q\}^\perp$.

Remark 3.2. Note that the binary relation $\perp$ on Σ is symmetric, anti-reflexive and *separating*, *i.e.*, for all $p \neq q \in \Sigma$, there is $r \in \Sigma$ such that $p \perp r$ and $q \not\perp r$.

Conversely, let Σ be a set and $\perp$ a symmetric, anti-reflexive and separating binary relation on Σ. Then $\mathcal{L} = \{A \subseteq \Sigma\ ;\ A^{\perp\perp} = A\}$ is an orthocomplemented simple closure space on Σ.

Definition 3.3 (D. Aerts, [1])**.** Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of orthocomplemented simple closure spaces on Σ_α, $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, and let p, $q \in \mathbf{\Sigma}$. Denote by $\#$ the binary relation on $\mathbf{\Sigma}$ defined by $p\#q$ if and only if there is $\beta \in \Omega$ such that $p_\beta \perp_\beta q_\beta$. Then

$$\underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha := \{R \subseteq \mathbf{\Sigma};\ R^{\#\#} = R\}\,.$$

Lemma 3.4. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of orthocomplemented simple closure spaces on Σ_α. Then*

$$\underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha = \textstyle\bigcircledwedge_\alpha \mathcal{L}_\alpha\,,$$

and $\underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha$ is orthocomplemented.

Proof. Let $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$. Obviously, $\#$ is symmetric and anti-reflexive. Since $\mathcal{L}_\alpha$ is orthocomplemented, $\perp_\alpha$ is separating. Therefore, it follows directly from Definition 3.3 that $\#$ is also separating. As a consequence, $\underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha$ is an orthocomplemented simple closure space on $\mathbf{\Sigma}$. Moreover, coatoms are given by $p^\# = \bigcup_\alpha \pi_\alpha^{-1}(p_\alpha^{\perp_\alpha})$. As a consequence, $\underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha \subseteq \bigcircledwedge_\alpha \mathcal{L}_\alpha$.

Let $a \in \prod_\alpha \mathcal{L}_\alpha$. Denote the set of coatoms of $\mathcal{L}_\alpha$ above a_α by $\Sigma'[a_\alpha]$. Then $a_\alpha = \bigcap_\alpha \Sigma'[a_\alpha]$. Moreover,

$$\bigcap \Big\{ \bigcup_\alpha \pi_\alpha^{-1}(x_\alpha)\ ;\ x \in \prod_\alpha \Sigma'[a_\alpha] \Big\} = \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)\,.$$

Hence $\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha) \in \underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha$. Another way to see this is to compute

$$\Big(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)\Big)^\# = \bigcap_\alpha (\pi_\alpha^{-1}(a_\alpha))^\# = \bigcap_\alpha \pi_\alpha^{-1}(a_\alpha^{\perp_\alpha}) = \prod_\alpha a_\alpha^{\perp_\alpha}\,,$$

whence

$$\Big(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)\Big)^{\#\#} = \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha^{\perp_\alpha\perp_\alpha}) = \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)\,.$$

As a consequence, $\bigcircledwedge_\alpha \mathcal{L}_\alpha \subseteq \underset{\text{Aerts}}{\bigotimes}{}_\alpha \mathcal{L}_\alpha$. $\square$

Remark 3.5. The symbol $\bigcircledwedge$ was originally used by Aerts.

3.2. The box product.

Definition 3.6. Let $\mathcal{L}$ be a lattice and $a \in \mathcal{L}$. We denote by $a\!\downarrow$ the set $\{x \in \mathcal{L}\ ;\ x \leq a\}$ and by L the set $\{a\!\downarrow\ ;\ a \in \mathcal{L}\}$ ordered by set-inclusion (note that $L \subseteq 2^{\mathcal{L}}$).

Definition 3.7 (G. Grätzer, F. Wehrung, [7])**.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices and $(a,b) \in \mathcal{L}_1 \times \mathcal{L}_2$. Define

$$a \,\square\, b = (a\!\downarrow \times \mathcal{L}_2) \cup (\mathcal{L}_1 \times b\!\downarrow)\,.$$

The *box product* $\mathcal{L}_1 \,\square\, \mathcal{L}_2$ is defined as the set of all finite intersections of the form $\bigcap\{a_i \,\square\, b_i\ ;\ i \leq n\}$ where $(a_i, b_i) \in \mathcal{L}_1 \times \mathcal{L}_2$ for all $i \leq n$, ordered by set-inclusion.

Remark 3.8. Obviously, $\mathcal{L}_1 \,\square\, \mathcal{L}_2$ is a meet-sublattice of $2^{\mathcal{L}_1 \times \mathcal{L}_2}$. In fact, $\mathcal{L}_1 \,\square\, \mathcal{L}_2$ is a lattice (see [7], Proposition 2.9). Note that if $\mathcal{L}_1$ and $\mathcal{L}_2$ have top elements, then $\mathcal{L}_1 \boxtimes \mathcal{L}_2 = \mathcal{L}_1 \,\square\, \mathcal{L}_2$ where $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is the *lattice tensor product* (see [7]).

Definition 3.9. Let Σ_1 and Σ_2 be nonempty sets, $\mathcal{L}_1 \subseteq 2^{\Sigma_1}$ and $\mathcal{L}_2 \subseteq 2^{\Sigma_2}$. Define $\mathcal{L}_1 \owedge_n \mathcal{L}_2$ as in Definition 2.9 but taking only finite intersections. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are atomistic lattices, then define $\mathcal{L}_1 \owedge_n \mathcal{L}_2$ as $l_1 \owedge_n l_2$, where $l_i = \{\Sigma[a_i]\ ;\ a_i \in \mathcal{L}_i\} \subseteq 2^{\Sigma_i}$ and $\Sigma[a_i]$ denotes the set of atoms under a_i ($\Sigma_i := \Sigma[1_i]$).

Proposition 3.10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be lattices. Then $\mathcal{L}_1 \,\square\, \mathcal{L}_2 = L_1 \owedge_n L_2$.*

Proof. Direct from Definitions 3.7 and 3.9. $\square$

Theorem 3.11. *For atomistic lattices, $\mathcal{L}_1 \owedge_n \mathcal{L}_2 \cong \mathcal{L}_1 \,\square\, \mathcal{L}_2$.*

Proof. Define $f\colon \mathcal{L}_1 \owedge_n \mathcal{L}_2 \to \mathcal{L}_1 \,\square\, \mathcal{L}_2$ as

$$f(\Sigma[a_1] \times \Sigma_2 \cup \Sigma_1 \times \Sigma[a_2]) = a_1 \,\square\, a_2\,,$$

and set $f(\bigcap x_i) := \bigcap f(x_i)$, where each x_i has the form $\Sigma[a_1] \times \Sigma_2 \cup \Sigma_1 \times \Sigma[a_2]$. Obviously, f is bijective and preserves meets, hence also joins. $\square$

3.3. The tensor products of Chu and Shmuely. We now compare $\ovee$ with three other tensor products of lattices appearing in the literature.

Definition 3.12 (P. H. Chu [2])**.** The category $\mathbf{Chu}(\mathbf{Set}, 2)$ has as objects triples $\mathsf{A} = (A, r, X)$ where A and X are sets and r is a map from $A \times X$ to 2 (where $2 = \{0, 1\}$). Arrows are pairs of maps $(F, G)\colon \mathsf{A} \to (B, s, Y)$ with $F\colon A \to B$ and $G\colon Y \to X$ such that $s(F(a), y) = r(a, G(y))$ for all $a \in A$ and $y \in Y$. There is a functor ${}^{\perp}$ and a bifunctor $\underset{\mathrm{Chu}}{\otimes}$ defined on objects as $\mathsf{A}^{\perp} = (X, \check{r}, A)$ with $\check{r}(x, a) = r(a, x)$, and $\mathsf{A} \underset{\mathrm{Chu}}{\otimes} \mathsf{B} = (A \times B, t, \mathbf{Chu}(\mathsf{A}, \mathsf{B}^{\perp}))$ with $t((a, b), (F, G)) = r(a, G(b)) = \check{s}(F(a), b)$.

Remark 3.13. A simple closure space $\mathcal{L} \in \mathbf{Cl}(\Sigma)$ can be identified with the Chu space $(\Sigma, r, \mathcal{L})$, which we also denote by $\mathcal{L}$, where $r(p, a) = 1$ if and only if $p \in a$.

Proposition 3.14. *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be simple closure spaces on Σ_0 and Σ_1 respectively. Then there is a one to one correspondence between invertible arrows in $\mathbf{Chu}(\mathcal{L}_0, \mathcal{L}_1)$ and isomorphisms between $\mathcal{L}_0$ and $\mathcal{L}_1$.*

Proof. Let $f\colon \mathcal{L}_0 \to \mathcal{L}_1$ be an isomorphism. Then $(f, f^{-1}) \in \mathbf{Chu}(\mathcal{L}_0, \mathcal{L}_1)$. Let (F, G) be an invertible arrow in $\mathbf{Chu}(\mathcal{L}_0, \mathcal{L}_1)$. Then $G\colon \mathcal{L}_1 \to \mathcal{L}_0$ is bijective, and since $G = F^{-1}$, G preserves arbitrary meets, hence also arbitrary joins. □

Definition 3.15. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be posets. A Galois connection (more precisely an adjunction) between $\mathcal{L}_1$ and $\mathcal{L}_2$ is a pair (f, g) of order-preserving maps with $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ and $g\colon \mathcal{L}_2 \to \mathcal{L}_1$, such that for all $a \in \mathcal{L}_1$ and $b \in \mathcal{L}_2$, $f(a) \le b \Leftrightarrow a \le g(b)$. Note that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete lattices, then there is a one to one correspondence between maps from $\mathcal{L}_1$ to $\mathcal{L}_2$ preserving arbitrary joins and adjunctions between $\mathcal{L}_1$ and $\mathcal{L}_2$.

The following is standard; we include the easy proof for completeness.

Lemma 3.16. *Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be complete join-semilattices and (f, g) a pair of maps with $f\colon \mathcal{L}_0 \to \mathcal{L}_1$ and $g\colon \mathcal{L}_1 \to \mathcal{L}_0$.*

(1) *If (f, g) forms a Galois connection, then f preserves arbitrary joins.*
(2) *Suppose that $\mathcal{L}_0$ is a simple closure space on Σ_0. For $b \in \mathcal{L}_1$, let $F^{-1}(b)$ denote the set $\{p \in \Sigma_0 \,;\, f(p) \le b\}$. Then f preserves arbitrary joins $\Leftrightarrow$ $f(a) = \bigvee\{f(p) \,;\, p \in a\}$ for all $a \in \mathcal{L}_0$ and $F^{-1}(b) \in \mathcal{L}_0$ for all $b \in \mathcal{L}_1$.*

Proof. (1) Let $\omega \subseteq \mathcal{L}_0$. Since f preserves order, $y := \bigvee\{f(x) \,;\, x \in \omega\} \le f(\bigvee \omega)$. On the other hand, for any $x \in \omega$, $f(x) \le y$, hence $x \le g(y)$. As a consequence, $\bigvee \omega \le g(y)$, therefore $f(\bigvee \omega) \le y$.

(2) ($\Rightarrow$) If $p \in \bigvee F^{-1}(b)$, then $f(p) \le \bigvee\{f(q) \,;\, q \in F^{-1}(b)\} \le b$, hence $p \in F^{-1}(b)$. ($\Leftarrow$) Obviously, (f, F^{-1}) forms a Galois connection between $\mathcal{L}_0$ and $\mathcal{L}_1$. □

Remark 3.17. Let $\mathcal{L}$ be a poset. Then $\mathcal{L}^*$ denotes the dual of $\mathcal{L}$ (defined by the converse order-relation) and $\le_*$ the order relation in $\mathcal{L}^*$.

Theorem 3.18. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be simple closure spaces on Σ_1 and Σ_2 respectively. Then*

(1) $\mathcal{L}_1 \circledvee \mathcal{L}_2 \cong \mathcal{L}_1 \underset{\mathrm{Chu}}{\otimes} \mathcal{L}_2$,
(2) *$(\mathcal{L}_1 \circledvee \mathcal{L}_2)^*$ is isomorphic to the set of join-preserving maps $\mathcal{L}_1 \to \mathcal{L}_2^*$, ordered point-wise (i.e., $f \le g$ if and only if $f(p_1) \le_* g(p_1)$ for all $p_1 \in \Sigma_1$).*

Proof. (1) We denote $\Sigma_1 \times \Sigma_2$ by $\mathbf{\Sigma}$. Let $(F, G) \in \mathbf{Chu}(\mathcal{L}_1, \mathcal{L}_2^{\perp})$. Define

$$R_{(F,G)} := \bigcup_{p_1 \in \Sigma_1} p_1 \times F(p_1)\,.$$

By definition, ${R_{(F,G)}}_2[p] \in \mathcal{L}_2$ for all $p \in \mathbf{\Sigma}$, and

$$q_1 \in {R_{(F,G)}}_1[p] \Leftrightarrow (q_1, p_2) \in R_{(F,G)} \Leftrightarrow p_2 \in F(q_1) \Leftrightarrow q_1 \in G(p_2)\,,$$

thus $R_{(F,G)_1}[p] \in \mathcal{L}_1$ for all $p \in \mathbf{\Sigma}$, hence $R_{(F,G)} \in \mathcal{L}_1 \circledvee \mathcal{L}_2$.

Let $R \in \mathcal{L}_1 \circledvee \mathcal{L}_2$. Define $F_R \colon \Sigma_1 \to \mathcal{L}_2$ and $G_R \colon \Sigma_2 \to \mathcal{L}_1$ as $F_R(p_1) := R_2[(p_1, \cdot)]$ and $G_R(p_2) := R_1[(\cdot, p_2)]$. Then

$$p_2 \in F_R(p_1) \Leftrightarrow (p_1, p_2) \in R \Leftrightarrow p_1 \in G_R(p_2)\,,$$

so that $(F_R, G_R) \in \mathbf{Chu}(\mathcal{L}_1, \mathcal{L}_2^{\perp})$. Note that $(F, G) \mapsto R_{(F,G)}$ is isotone in both directions.

Obviously,

$$F_{R_{(F,G)}} = F,\ G_{R_{(F,G)}} = G, \text{ and } R_{(F_R, G_R)} = R\,.$$

As a consequence, there is an invertible arrow in $\mathbf{Chu}(\mathcal{L}_1 \circledvee \mathcal{L}_2, \mathcal{L}_1 \underset{\mathrm{Chu}}{\otimes} \mathcal{L}_2)$.

(2) Let $(F, G) \in \mathbf{Chu}(\mathcal{L}_1, \mathcal{L}_2^{\perp})$. Define $f \colon \mathcal{L}_1 \to \mathcal{L}_2^*$ and $g \colon \mathcal{L}_2^* \to \mathcal{L}_1$ as $f(a) := \bigcap F(a)$ and $g(b) = \bigcap G(b)$. Then obviously (f, g) forms a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2^*$.

Let $f \colon \mathcal{L}_1 \to \mathcal{L}_2^*$ be $\bigvee$-preserving. Define $g \colon \mathcal{L}_2 \to \mathcal{L}_1$ as $g(b) := \bigvee\{a \in \mathcal{L}_1\ ; f(a) \leq_* b\}$. Then obviously $(f, g) \in \mathbf{Chu}(\mathcal{L}_1, \mathcal{L}_2^{\perp})$. $\square$

Remark 3.19. Note that $\circledvee$ is the $\boxtimes$-tensor product of Golfin [6]. As a corollary of Theorem 3.18 part (2), $\mathcal{L}_1 \circledvee \mathcal{L}_2 \cong \mathcal{L}_1 \otimes \mathcal{L}_2$ the tensor product of Shmuely [12].

3.4. The semilattice tensor product of Fraser. We now consider tensor products as solutions to universal mapping problems.

Definition 3.20. Let $\mathbf{C}$ be a concrete category over $\mathbf{Set}$. Let $\{\mathsf{A}_\alpha\}_{\alpha \in \Omega}$ and C be objects of $\mathbf{C}$. A *multimorphism* of $\mathbf{C}$ is an arrow $f \in \mathbf{Set}(\prod_\alpha \mathsf{A}_\alpha, \mathsf{C})$ such that for all $a \in \prod_\alpha \mathsf{A}_\alpha$, $f(a[-, \alpha]) \in \mathbf{C}(\mathsf{A}_\alpha, \mathsf{C})$ (see Definition 2.3). If $|\Omega| = 2$, then f is called a *bimorphism*.

Definition 3.21 (G. Seal, [11])**.** Let $\mathbf{C}$ be a concrete category over $\mathbf{Set}$. A *tensor product* in $\mathbf{C}$ is a bifunctor $- \otimes - \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ such that for any objects A, B of $\mathbf{C}$ there is a a bimorphism $f \colon \mathsf{A} \times \mathsf{B} \to \mathsf{A} \otimes \mathsf{B}$, and for any object C of $\mathbf{C}$ and any bimorphism $g \colon \mathsf{A} \times \mathsf{B} \to \mathsf{C}$ there is a unique arrow $h \in \mathbf{C}(\mathsf{A} \otimes \mathsf{B}, \mathsf{C})$ which makes the diagram

$$\begin{tikzcd}
\mathsf{A} \times \mathsf{B} \arrow[r, "f"] \arrow[d, "g"'] & \mathsf{A} \otimes \mathsf{B} \arrow[dl, "!h"] \\
\mathsf{C} &
\end{tikzcd}$$

commute.

Remark 3.22. By definition, the tensor product is unique up to isomorphisms. Consider now the category $\mathbf{C}$ of join-semilattices with maps preserving finite joins. Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ be join-semilattices. If $\mathcal{L}$ is a tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbf{C}$, then $f(\mathcal{L}_1 \times \mathcal{L}_2)$ generates $\mathcal{L}$. Indeed, let $\mathcal{L}_0$ be a (join-semilattice) tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$ as defined by Fraser in [5] (write g the bimorphism from $\mathcal{L}_1 \times \mathcal{L}_2$ to $\mathcal{L}_0$). Then by definition $\mathcal{L}_0$ is generated by $g(\mathcal{L}_1 \times \mathcal{L}_2)$, and obviously $\mathcal{L}_0$ is a tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbf{C}$. Therefore there is an isomorphism $h\colon \mathcal{L}_0 \to \mathcal{L}$ such that $f = h \circ g$. Hence, for every $x \in \mathcal{L}$ we have $x = h(h^{-1}(x)) = h(\bigvee g(a_i, b_i)) = \bigvee h(g(a_i, b_i)) = \bigvee f(a_i, b_i)$ where $a_i \in \mathcal{L}_1$ and $b_i \in \mathcal{L}_2$.

As a consequence, for the category of join-semilattices with maps preserving finite joins, Definition 3.21 is equivalent to the definition of Fraser.

Theorem 3.23. $\circledvee$ *is a tensor product in* $\mathbf{Cl}$.

Proof. Let $\mathcal{L}_1$, $\mathcal{L}_2 \in \mathbf{Cl}$. By Theorem 2.12, $\mathcal{L}_1 \circledvee \mathcal{L}_2 \in \mathbf{Cl}$, and by Lemma 2.7.3, the map $f\colon \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_1 \circledvee \mathcal{L}_2$ sending $a = (a_1, a_2)$ to $\overline{a} = a_1 \times a_2$ is a bimorphism.

Let $\mathcal{L} \in \mathbf{Cl}$ (or a complete join-semilattice), and let $g\colon \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}$ be a bimorphism. Define $h\colon \mathcal{L}_1 \circledvee \mathcal{L}_2 \to \mathcal{L}$ by $h(R) = \bigvee\{g(p_1, p_2)\ ;\ (p_1, p_2) \in R\}$. Furthermore, for $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$, define $g_{p_1}\colon \mathcal{L}_2 \to \mathcal{L}$ and $g_{p_2}\colon \mathcal{L}_1 \to \mathcal{L}$ as $g_{p_1}(y) := g(p_1, y)$ and $g_{p_2}(x) := g(x, p_2)$ respectively. Let $b \in \mathcal{L}$. Write, as in Lemma 3.16, $H^{-1}(b) = \{p \in \Sigma_1 \times \Sigma_2\ ;\ h(p) \leq b\}$, $G_{p_1}^{-1}(b) = \{s \in \Sigma_2\ ;\ g_{p_1}(s) \leq b\}$, and $G_{p_2}^{-1}(b) = \{r \in \Sigma_1\ ;\ g_{p_2}(r) \leq b\}$. Then

$$H^{-1}(b) = \bigcup_{p_1 \in \Sigma_1} p_1 \times G_{p_1}^{-1}(b) = \bigcup_{p_2 \in \Sigma_2} G_{p_2}^{-1}(b) \times p_2\,.$$

Since g is a bimorphism, it follows from Lemma 3.16 and Definition 2.9 that $H^{-1}(b) \in \mathcal{L}_1 \circledvee \mathcal{L}_2$. Therefore, by Lemma 3.16, $h \in \mathbf{Cl}(\mathcal{L}_1 \circledvee \mathcal{L}_2, \mathcal{L})$. Finally, if $h' \in \mathbf{Cl}(\mathcal{L}_1 \circledvee \mathcal{L}_2, \mathcal{L})$ and $h' \circ f = g$, then h' equals h on atoms, therefore $h' = h$.

Let L_1, $L_2 \in \mathbf{Cl}$, $f_1 \in \mathbf{Cl}(\mathcal{L}_1, L_1)$, and $f_2 \in \mathbf{Cl}(\mathcal{L}_2, L_2)$. Then $g = f \circ (f_1 \times f_2)\colon \mathcal{L}_1 \times \mathcal{L}_2 \to L_1 \circledvee L_2$ is a bimorphism. We define $f_1 \circledvee f_2 \in \mathbf{Cl}(\mathcal{L}_1 \circledvee \mathcal{L}_2, L_1 \circledvee L_2)$ to be the arrow h constructed above. $\square$

Remark 3.24. Let $\mathbf{C}_{\mathrm{com}}$ denote the category of complete join-semilattices with maps preserving arbitrary joins. Let $\mathcal{L}_1$ and $\mathcal{L}_2 \in \mathbf{Cl}$. Then $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 \circledvee \mathcal{L}_2 \in \mathbf{C}_{\mathrm{com}}$. Moreover, we have proved that $\mathcal{L}_1 \circledvee \mathcal{L}_2$ is the tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathbf{C}_{\mathrm{com}}$.

4. Equivalent definition

In this section we give an equivalent definition of $\mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$ and $\mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$ in terms of a universal property with respect to a given class of bimorphisms of $\mathbf{Cl}$.

Lemma 4.1. *Let $\mathcal{L}$ be a simple closure space on Σ. For all $a \in \mathcal{L}$, there is a unique mapping $h_a\colon \mathcal{L} \to 2$ preserving arbitrary joins such that for all $p \in \Sigma$, $h_a(p) = 0$ if and only if $p \in a$.*

Proof. Let $b \in \mathcal{L}$. Define h_a by $h_a(b) = \bigvee\{h_a(p) ; p \in b\}$ (hence $h_a(0) = 0$) with $h_a(p) = 0$ if $p \in a$ and $h_a(p) = 1$ if $p \notin a$. Then, by Lemma 3.16, h preserves arbitrary joins. $\square$

Lemma 4.2. *Let $\mathcal{L}$ and $\mathcal{L}_\alpha$ $(\alpha \in \Omega)$ be simple closure spaces. Then*

(1) *If $f\colon \prod_\alpha \mathcal{L}_\alpha \to \mathcal{L}$ is a multimorphism of $\mathbf{Cl}$ and $v \in \prod_\alpha \mathsf{Aut}(\mathcal{L}_\alpha)$, then $f \circ v$ is again a multimorphism.*
(2) *For $g \in \prod_\alpha \mathbf{Cl}(\mathcal{L}_\alpha, 2)$, the mapping $\bigcap_\alpha g_\alpha\colon \prod_\alpha \mathcal{L}_\alpha \to 2$, defined as*

$$\bigcap_\alpha g_\alpha(a) := \bigcap\{g_\alpha(a_\alpha) ; \alpha \in \Omega\},$$

is a multimorphism.

Proof. (1) Let $w = f \circ v$, $a \in \prod_\alpha \mathcal{L}_\alpha$, $\beta \in \Omega$, and $\omega \subseteq \mathcal{L}_\beta$. Then

$$w(a[\bigvee \omega, \beta]) = f(v(a)[\bigvee\{v_\beta(x) ; x \in \omega\}, \beta]) = \bigvee_{x \in \omega} w(a[x, \beta]),$$

since f is a multimorphism.

(2) Let $g = \bigcap_\alpha g_\alpha$, $a \in \prod_\alpha \mathcal{L}_\alpha$, $\beta \in \Omega$, and $b \in \mathcal{L}_\beta$. Write

$$h(-) := g(a[-, \beta])\colon \mathcal{L}_\beta \to 2.$$

Then $h(b) = g_\beta(b) \cap_{\alpha \neq \beta} g_\alpha(a_\alpha)$. Hence $h(b) = 0$ if $g_\alpha(a_\alpha) = 0$ for some $\alpha \neq \beta$ and $h(b) = g_\beta(b)$ otherwise. As a consequence, h preserves arbitrary joins, hence g is a multimorphism of $\mathbf{Cl}$. $\square$

Definition 4.3. Let $\mathcal{L}$ and $\mathcal{L}_\alpha$ $(\alpha \in \Omega)$ be simple closure spaces, $f\colon \prod_\alpha \mathcal{L}_\alpha \to \mathcal{L}$ a multimorphism of $\mathbf{Cl}$ and $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$. Put $T := \prod_\alpha T_\alpha$. Then we define

$$V_T(f) := \{f \circ v\colon \prod_\alpha \mathcal{L}_\alpha \to \mathcal{L} ; v \in T\},$$
$$\Gamma_\cap := \{\bigcap_\alpha g_\alpha\colon \prod_\alpha \mathcal{L}_\alpha \to 2 ; g \in \prod_\alpha \mathbf{Cl}(\mathcal{L}_\alpha, 2)\}.$$

Definition 4.4. Let $\mathcal{L}$ be a simple closure space and $A \subseteq \mathcal{L}$. We say that A *generates* $\mathcal{L}$ if each element of $\mathcal{L}$ is the join of elements in A.

Theorem 4.5. *Let $\{\mathcal{L}_\alpha\}_{\alpha \in \Omega}$ be a family of simple closure spaces on Σ_α, $\mathcal{L}$ a simple closure space on $\mathbf{\Sigma}$, and $T = \prod_\alpha T_\alpha$ with $T_\alpha \subseteq \mathsf{Aut}(\mathcal{L}_\alpha)$.*

(1) *If $\mathcal{L} \in \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$, then there is a multimorphism $f\colon \prod_\alpha \mathcal{L}_\alpha \to \mathcal{L}$ of $\mathbf{Cl}$ such that $f(\prod_\alpha \mathcal{L}_\alpha)$ generates $\mathcal{L}$, and for all $g \in \Gamma_\cap$ and $w \in V_T(f)$, there*

is a unique $h \in \mathbf{Cl}(\mathcal{L}, 2)$ and a unique $u \in \mathbf{Cl}(\mathcal{L}, \mathcal{L})$ such that the following diagrams commute.

$$\begin{tikzcd} \prod_\alpha \mathcal{L}_\alpha \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{L} \arrow[dl, "!h"] \\ 2 & \end{tikzcd} \qquad \begin{tikzcd} \prod_\alpha \mathcal{L}_\alpha \arrow[r, "f"] \arrow[d, "w"'] & \mathcal{L} \arrow[dl, "!u"] \\ \mathcal{L} & \end{tikzcd}$$

(2) *Conversely, if there is a multimorphism of* **Cl** $f\colon \prod_\alpha \mathcal{L}_\alpha \to \mathcal{L}$ *satisfying all conditions of part* (1)*, then there is some* $\mathcal{L}_0 \in \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$ *such that* $\mathcal{L} \cong \mathcal{L}_0$.

Proof. (1) By Lemma 2.7.3, the map $f\colon \prod_\alpha \mathcal{L}_\alpha \to \mathcal{L}$ sending a to $\overline{a}$, is a multimorphism of **Cl**, and obviously, $f(\prod_\alpha \mathcal{L}_\alpha)$ generates $\mathcal{L}$. Moreover, by Axiom P4, for all $w \in V_T(f)$ there is a unique u such that the second diagram commutes.

Let $g = \bigcap_\alpha g_\alpha \in \Gamma_\cap$. Write G_α for the restriction to atoms of g_α. Let $a_\alpha := \bigvee G_\alpha^{-1}(0)$. Note that by Lemma 3.16, $a_\alpha = G_\alpha^{-1}(0)$. Recall that by hypothesis, $\mathcal{L} \in \mathbf{Cl}(\mathbf{\Sigma})$ with $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$. Define $x := \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)$. By Axiom P2 $x \in \mathcal{L}$, and by Lemma 4.1 there is $h_x \in \mathbf{Cl}(\mathcal{L}, 2)$ such that for all $p \in \mathbf{\Sigma}$, $h_x(p) = 0$ if and only if $p \in x$. Hence, $h_x \circ f = g$. Let $h \in \mathbf{Cl}(\mathcal{L}, 2)$ such that $h \circ f = g$. Then on atoms h equals h_x, therefore $h = h_x$.

(2) Let $\mathbf{\Sigma_0} = \prod_\alpha \Sigma_\alpha$. We denote by F the mapping from $\mathbf{\Sigma_0}$ to $\mathcal{L}$ induced by the multimorphism f.

(2.1) For $a, b \in \prod_\alpha \mathcal{L}_\alpha \backslash \{0_\alpha\}$, we write $a \le b$ if and only if $a_\beta \le b_\beta$, for all $\beta \in \Omega$ (this is the standard product ordering on $\prod_\alpha \mathcal{L}_\alpha$). *Claim*: $f(a) \subseteq f(b) \Rightarrow a \le b$. As a corollary, f is injective. [Proof: Suppose that $f(a) \subseteq f(b)$ and that $a_\beta \not\le b_\beta$ for some $\beta \in \Omega$. By Lemma 4.1, there is $h_{b_\beta} \in \mathbf{Cl}(\mathcal{L}_\beta, 2)$ such that for all $p \in \Sigma_\beta$, $h_{b_\beta}(p) = 0$ if and only if $p \in b_\beta$. Let $g_\beta := h_{b_\beta}$, $g_\alpha := h_{0_\alpha}$ for all $\alpha \ne \beta$, and $g := \bigcap_\alpha g_\alpha$. By definition, $g \in \Gamma_\cap$, hence there is $h \in \mathbf{Cl}(\mathcal{L}, 2)$ such that $h \circ f = g$. As a consequence,

$$1 = g_\beta(a_\beta) = g(a) = h(f(a)) \subseteq h(f(b)) = g(b) = g_\beta(b_\beta) = 0\,,$$

a contradiction. This proves the claim.]

As a consequence, since $f(\prod_\alpha \mathcal{L}_\alpha)$ generates $\mathcal{L}$, for all $p \in \mathbf{\Sigma_0}$, $F(p)$ is an atom of $\mathcal{L}$, and the mapping from $\mathbf{\Sigma_0}$ to $\mathbf{\Sigma}$ induced by F (which we also denote by F) is bijective. Moreover, for all $a \in \prod_\alpha \mathcal{L}_\alpha$,

$$f(a) = \bigvee \{F(p)\ ;\ p \in \overline{a}\}\,.$$

Therefore, for all $a, b \in \prod_\alpha \mathcal{L}_\alpha$, we have: $a \le b \Rightarrow f(a) \subseteq f(b)$ (note that if Ω is finite, then this implication follows directly from the fact that f is a multimorphism).

(2.2) Let $\mathcal{L}_0 \subseteq 2^{\mathbf{\Sigma_0}}$ defined as $\mathcal{L}_0 := \{F^{-1}(c) \;;\; c \in \mathcal{L}\}$. Then by what precedes, $\mathcal{L}_0 \in \mathbf{Cl}(\mathbf{\Sigma_0})$ and the map $F^{-1}\colon \mathcal{L} \to \mathcal{L}_0$ is bijective and preserves arbitrary meets, hence also arbitrary joins. It remains to prove that $\mathcal{L}_0 \in \mathcal{S}_T(\mathcal{L}_\alpha, \alpha \in \Omega)$, hence to check that Axioms P2, P3 and P4 hold in $\mathcal{L}_0$. Below, if $g \in \Gamma_\cap$, then G denotes the map from $\mathbf{\Sigma_0}$ to 2 induced by g.

(P2) Let $a \in \prod_\alpha \mathcal{L}_\alpha$, $x := F(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha))$, and $p \in \mathbf{\Sigma_0}$ such that $p_\alpha \notin a_\alpha$ for all $\alpha \in \Omega$. Suppose that $F(p) \in \bigvee x$. From Lemma 4.1, for all $\beta \in \Omega$, there is $h_{a_\beta} \in \mathbf{Cl}(\mathcal{L}_\beta, 2)$ such that for all $p \in \Sigma_\beta$, $h_{a_\beta}(p) = 0$ if and only if $p \in a_\beta$. Let $g = \bigcap_\alpha h_{a_\alpha}$. By definition, $g \in \Gamma_\cap$, hence there is $h \in \mathbf{Cl}(\mathcal{L}, 2)$ such that $h \circ f = g$; whence

$$\begin{aligned} 1 = G(p) = h(F(p)) \subseteq h(\bigvee x) &= \bigvee\{h(F(p)) \;;\; p \in \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)\} \\ &= \bigvee\{G(p) \;;\; p \in \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)\} = 0\,, \end{aligned}$$

a contradiction. As a consequence, $F^{-1}(\bigvee F(\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha))) = \bigcup_\alpha \pi_\alpha^{-1}(a_\alpha)$, hence $\bigcup_\alpha \pi_\alpha^{-1}(a_\alpha) \in \mathcal{L}_0$.

(P3) Let $\beta \in \Omega$, $A \subseteq \Sigma_\beta$, and $p \in \mathbf{\Sigma_0}$, such that $p[A, \beta] \in \mathcal{L}_0$, *i.e.*, there is $c \in \mathcal{L}$ such that $p[A, \beta] = F^{-1}(c)$. Let $q \in \bigvee A$ and $\hat{p} \in \prod_\alpha \mathcal{L}_\alpha$ such that $\overline{\hat{p}} = \{p\}$. Then $p[q, \beta] \in p[\bigvee A, \beta]$, hence, since f is multimorphism, we find that

$$F(p[q,\beta]) \in F(p[\bigvee A, \beta]) \subseteq \bigvee F(p[\bigvee A, \beta]) = f(\hat{p}[\bigvee A, \beta]) = \bigvee_{q \in A} F(p[q, \beta]) \subseteq c\,.$$

As a consequence, $q \in A$, therefore $A \in \mathcal{L}_\beta$.

(P4) Let $v \in T$ and $u \in \mathbf{Cl}(\mathcal{L}, \mathcal{L})$ such that $u \circ f = f \circ v$. Define $u_0 := F^{-1} \circ u \circ F$. Then $u_0 \in \mathsf{Aut}(\mathcal{L}_0)$ and $u_0(p)_\alpha = v_\alpha(p_\alpha)$ for all $p \in \mathbf{\Sigma_0}$ and all $\alpha \in \Omega$. □

5. Central elements

Let $\{\mathcal{L}_\alpha\}_{\alpha \in \Omega}$ be a family of simple closure spaces and $\beta \in \Omega$. In this section we prove that if z is a central element of $\mathcal{L}_\beta$, then $\pi_\beta^{-1}(z)$ is a central element of $\textcircled{\vee}_\alpha \mathcal{L}_\alpha$ and of $\textcircled{\wedge}_\alpha \mathcal{L}_\alpha$. As a corollary, if $\textcircled{\vee}_\alpha \mathcal{L}_\alpha$ or if $\textcircled{\wedge}_\alpha \mathcal{L}_\alpha$ is irreducible, then all $\mathcal{L}_\alpha$'s are irreducible. We give some sufficient conditions under which the converse result holds in $\textcircled{\wedge}_\alpha \mathcal{L}_\alpha$.

Definition 5.1. Let a and b be elements of a lattice. Then (a, b) is said to be a *modular pair* (in symbols $(a, b)M$) if $(c \vee a) \wedge b = c \vee (a \wedge b)$ for all $c \leq b$.

Lemma 5.2. *Let $\mathcal{L}$ be a simple closure space on Σ and $z \in \mathcal{L}$. Then z is a central element of $\mathcal{L}$ if and only if $z^c := \Sigma \backslash z \in \mathcal{L}$ and $(z, z^c)M$ and $(z^c, z)M$.*

Proof. Direct from Theorem 4.13 (ε) in [10]. □

Corollary 5.3. *Let $\mathcal{L}$ be a simple closure space on Σ, z a central element, $a \subseteq z$, and $b \subseteq z^c$. Then $a \vee b = a \cup b$.*

Proof. From $(z^c, z)M$ follows that $(a \vee b) \cap z \subseteq (a \vee z^c) \cap z = a$, and from $(z, z^c)M$ follows that $(a \vee b) \cap z^c \subseteq (b \vee z) \cap z^c = b$. □

Theorem 5.4. *Let $\{\mathcal{L}_\alpha\}_{\alpha \in \Omega}$ be a family of simple closure spaces on Σ_α and $\beta \in \Omega$. If z is a central element of $\mathcal{L}_\beta$, then $\pi_\beta^{-1}(z)$ is a central element of $\textcircled{\vee}_\alpha \mathcal{L}_\alpha$.*

Proof. We denote $\pi_\beta^{-1}(z)$ by Z. Hence $Z^c = \pi_\beta^{-1}(z^c)$. From Lemma 5.2, $z^c := \Sigma_\beta \backslash z \in \mathcal{L}_\beta$. Therefore, by Axiom P2, $Z^c \in \textcircled{\vee}_\alpha \mathcal{L}_\alpha$.

We now prove that $(Z, Z^c)M$. The proof for $(Z^c, Z)M$ is similar. Let $R \in \textcircled{\vee}_\alpha \mathcal{L}_\alpha$ with $R \subseteq Z^c$. Let $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, $p \in \mathbf{\Sigma}$, $X = R \cup Z$, and $\alpha \in \Omega$. If $\alpha \neq \beta$, then $X_\alpha[p] = \Sigma_\alpha$ if $p_\beta \in z$ and $X_\alpha[p] = R_\alpha[p]$ otherwise. On the other hand, $X_\beta[p] = z \cup R_\beta[p]$, hence $X_\beta[p] \in \mathcal{L}_\beta$ by Corollary 5.3. As a consequence, $R \vee Z = R \cup Z$, therefore $(R \vee Z) \cap Z^c = (R \cup Z) \cap Z^c = R$. □

Theorem 5.5 ([9], Theorem 1)**.** *Let $\{\mathcal{L}_\alpha\}_{\alpha \in \Omega}$ be a family of orthocomplemented simple closure spaces on Σ_α. Suppose that one of the following assumptions holds.*

(1) *Ω is finite.*
(2) *For all $\alpha \in \Omega$, $\mathcal{L}_\alpha$ has the covering property, and for all $p \neq q \in \Sigma_\alpha$ having the same central cover, $p \vee q$ contains an infinite number of atoms.*

Then $\textcircled{\wedge}_\alpha \mathcal{L}_\alpha$ is irreducible if and if all $\mathcal{L}_\alpha$'s are irreducible.

6. Automorphisms

In this section we prove the following result. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be simple closure spaces different from 2, $\mathcal{L} \in \mathsf{S}(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ and $u \in \mathbf{Cl}(\mathcal{L}, \mathcal{L})$ sending atoms to atoms. If u is *large* (see Definition 6.3 below), then there is a permutation f of $\{1, \ldots, n\}$ and arrows $v_i \in \mathbf{Cl}(\mathcal{L}_i, \mathcal{L}_{f(i)})$ sending atoms to atoms such that for any atom p of $\mathcal{L}$, $u(p)_{f(i)} = v_i(p_i)$. We need some hypotheses on each $\mathcal{L}_i$ which are true for instance if each $\mathcal{L}_i$ is irreducible orthocomplemented with the covering property or an irreducible DAC-lattice. Note that our hypotheses imply irreducibility. Note also that if u is an automorphism, then u is large.

Definition 6.1. Let $\mathcal{L}$ be a simple closure space on Σ. We say that $\mathcal{L}$ is *weakly connected* if $\mathcal{L} \neq 2$ and if there is a *connected covering* of Σ, that is a family of subsets $\{A^\gamma \subseteq \Sigma \,;\, \gamma \in \sigma\}$ such that

(1) $\Sigma = \bigcup\{A^\gamma \,;\, \gamma \in \sigma\}$ and $|A^\gamma| \geq 2$ for all $\gamma \in \sigma$,
(2) for all $\gamma \in \sigma$ and all $p \neq q \in A^\gamma$, $p \vee q$ contains a third atom,
(3) for all p, $q \in \Sigma$, there is a finite subset $\{\gamma_1, \ldots, \gamma_n\} \subseteq \sigma$ such that $p \in A^{\gamma_1}$, $q \in A^{\gamma_n}$, and such that $|A^{\gamma_i} \cap A^{\gamma_{i+1}}| \geq 2$ for all $1 \leq i \leq n-1$.

We say that $\mathcal{L}$ is *connected* if $\mathcal{L} \neq 2$ and for all $p, q \in \Sigma$, $p \vee q$ contains a third atom, say r, such that $p \in q \vee r$ and $q \in p \vee r$.

Remark 6.2. Note that in part (2) of Definition 6.1, it is not required that the third atom under $p \vee q$ is in A^γ. Note also that by Corollary 5.3, weakly connected implies irreducible. Finally, let $\mathcal{L}$ be a simple closure space. If $\mathcal{L} \neq 2$ and $\mathcal{L}$ is irreducible orthocomplemented with the covering property or an irreducible DAC-lattice, then $\mathcal{L}$ is connected.

Definition 6.3. Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α different from 2. Let $\mathcal{L} \in \mathsf{S}(\mathcal{L}_\alpha, \alpha \in \Omega)$, $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, and let $u \in \mathbf{Cl}(\mathcal{L}, \mathcal{L})$. We say that u is *large* if for all $\beta \in \Omega$ and $p \in \mathbf{\Sigma}$, $u(p[\Sigma_\beta])$ is not an atom of $\mathcal{L}$, and $u(1) \nsubseteq \pi_\beta^{-1}(p_\beta)$.

Lemma 6.4. *Let $\{\mathcal{L}_\alpha\}_{\alpha\in\Omega}$ be a family of simple closure spaces on Σ_α, $\mathbf{\Sigma} = \prod_\alpha \Sigma_\alpha$, and let $\mathcal{L}$ be a simple closure space on $\mathbf{\Sigma}$. Suppose that Axiom* P2 *holds in $\mathcal{L}$. Let $p, q \in \mathbf{\Sigma}$.*

(1) *If $p_\beta \neq q_\beta$ for at lest two $\beta \in \Omega$, then $p \vee q = p \cup q$.*
(2) *For all $\beta \neq \gamma \in \Omega$ and for all $b \in \mathcal{L}_\beta$ and $c \in \mathcal{L}_\gamma$ such that $p_\beta \in b$ and $p_\gamma \in c$, $p[b,\beta] \vee p[c,\gamma] = p[b,\beta] \cup p[c,\gamma]$.*

Proof. (1) Let $r \in p \vee q$, and suppose that $p_\beta \neq q_\beta$ and $p_\gamma \neq q_\gamma$ for some $\beta \neq \gamma \in \Omega$. We must show that $r \in p \cup q$, *i.e.*, that $r = p$ or $r = q$. Now

$$\begin{aligned} p,\, q \in (\pi_\beta^{-1}(p_\beta) \cup \pi_\gamma^{-1}(q_\gamma)) \cap (\pi_\beta^{-1}(q_\beta) \cup \pi_\gamma^{-1}(p_\gamma)) \\ = (\pi_\beta^{-1}(p_\beta) \cap \pi_\gamma^{-1}(p_\gamma)) \cup (\pi_\beta^{-1}(q_\beta) \cap \pi_\gamma^{-1}(q_\gamma))\,, \end{aligned}$$

because

$$\pi_\beta^{-1}(p_\beta) \cap \pi_\beta^{-1}(q_\beta) = \emptyset = \pi_\gamma^{-1}(p_\gamma) \cap \pi_\gamma^{-1}(q_\gamma)\,,$$

since by hypothesis, $p_\beta \neq q_\beta$ and $p_\gamma \neq q_\gamma$.

Now, by Axiom P2, $\pi_\alpha^{-1}(x_\alpha) \in \mathcal{L}$ for all $x \in \prod_\alpha \mathcal{L}_\alpha$ and all $\alpha \in \Omega$. Therefore,

$$(\pi_\beta^{-1}(p_\beta) \cup \pi_\gamma^{-1}(q_\gamma)) \cap (\pi_\beta^{-1}(q_\beta) \cup \pi_\gamma^{-1}(p_\gamma)) \in \mathcal{L}\,.$$

As a consequence,

$$p \vee q \subseteq (\pi_\beta^{-1}(p_\beta) \cap \pi_\gamma^{-1}(p_\gamma)) \cup (\pi_\beta^{-1}(q_\beta) \cap \pi_\gamma^{-1}(q_\gamma))\,.$$

It follows that either $r_\beta = p_\beta$ and $r_\gamma = p_\gamma$, or $r_\beta = q_\beta$ and $r_\gamma = q_\gamma$. Since this holds for every pair of indices β, γ at which p and q differ, either $r = p$ or $r = q$.

(2) Let $\beta \neq \gamma \in \Omega$, $b \in \mathcal{L}_\beta$, $c \in \mathcal{L}_\gamma$, and $p, q \in \mathbf{\Sigma}$ such that $p_\beta \in b$ and $p_\gamma \in c$. We must show that $p[c,\gamma] \cup p[b,\beta] \in \mathcal{L}$. Now, since $p_\beta \in b$ and $p_\gamma \in c$, we have,

$$\bigcap_{\alpha\neq\beta,\gamma} \pi_\beta^{-1}(b) \cap \pi_\gamma^{-1}(c) \cap \left(\pi_\beta^{-1}(p_\beta) \cup \pi_\gamma^{-1}(p_\gamma)\right) \cap \pi_\alpha^{-1}(p_\alpha) = p[c,\gamma] \cup p[b,\beta]\,.$$

To conclude, it suffices to note that by Axiom P2, the subsets of $\mathbf{\Sigma}$ $\pi_\beta^{-1}(b)$, $\pi_\gamma^{-1}(c)$, $\pi_\beta^{-1}(p_\beta) \cup \pi_\gamma^{-1}(p_\gamma)$, and $\pi_\alpha^{-1}(p_\alpha)$ are elements of $\mathcal{L}$. $\square$

Theorem 6.5. *Let Ω be a finite set and $\{\mathcal{L}_i\}_{i\in\Omega}$ a finite family of connected simple closure spaces on Σ_i. Let $\mathcal{L} \in \mathsf{S}(\mathcal{L}_i, i \in \Omega)$ and $u \in \mathbf{Cl}(\mathcal{L}, \mathcal{L})$ large, sending atoms to atoms. Then there is a bijection f of Ω, and for each $i \in \Omega$, there is $v_i \in \mathbf{Cl}(\mathcal{L}_i, \mathcal{L}_{f(i)})$ sending atoms to atoms such that $u(p)_{f(i)} = v_i(p_i)$ for all $p \in \mathbf{\Sigma}$ and $i \in \Omega$.*

Proof. The proof is similar to the proof of Theorem 3 in [9].

(1) Let $p \in \mathbf{\Sigma}$ and $j \in \Omega$. *Claim*: There is $k \in \Omega$ such that

$$u(p[\Sigma_j]) \subseteq u(p)[\Sigma_k]\,.$$

[Proof: Since $\mathcal{L}_j$ is connected, for all $q_j \in \Sigma_j$ different from p_j, $p_j \vee q_j$ contains a third atom, say r_j, and $p_j \in q_j \vee r_j$ and $q_j \in p_j \vee r_j$. Suppose that $u(p)_k \neq u(p[q_j])_k$ for at lest two indices k. Then, by Lemma 6.4 part (1) and Lemma 2.7.3,

$$u(p[r_j]) \in u(p \vee p[q_j]) = u(p) \vee u(p[q_j]) = u(p) \cup u(p[q_j])\,.$$

Assume for instance that $u(p[r_j]) = u(p)$. Then

$$u(p[q_j]) \in u(p \vee p[r_j]) = u(p) \vee u(p[r_j]) = u(p)\,,$$

a contradiction, which proves the claim.]

(2) Let $j \in \Omega$. Denote the k of part (1) by $f(j,p)$. *Claim*: The map $p \mapsto f(j,p)$ is constant. [Proof: Let p, $q \in \mathbf{\Sigma}$ differ only by one component, say $j' \neq j$, *i.e.*, $p_{j'} \neq q_{j'}$, and $p_i = q_i$, for all $i \neq j'$. Suppose that $f(j,p) \neq f(j,q)$. Write $k := f(j,p)$ and $k' := f(j,q)$.

(2.1) We first prove that

$$u(p[\Sigma_j]) \vee u(q[\Sigma_j]) = u(p[\Sigma_j]) \cup u(q[\Sigma_j])\,.$$

By hypothesis,

$$u(p[\Sigma_j]) \subseteq u(p)[\Sigma_k] \text{ and } u(q[\Sigma_j]) \subseteq u(q)[\Sigma_{k'}]\,.$$

Hence, for all $r_j \in \Sigma_j$, we have

$$u(p)_l = u(p[r_j])_l \ \forall l \neq k \text{ and } u(q)_m = u(q[r_j])_m \ \forall m \neq k'\,.$$

Since u is large, $u(p[\Sigma_j])$ is not an atom, hence there is $r_j \in \Sigma_j$ such that

$$u(p[r_j])_k \neq u(q)_k\,.$$

Note that $u(q)_k = u(q[r_j])_k$. Therefore, since $p[r_j]$ and $q[r_j]$ differ only by one component (namely j'), by part (1) we have

$$u(p[r_j])_l = u(q[r_j])_l \ \forall l \neq k\,.$$

As a consequence,

$$u(q)_l = u(q[r_j])_l = u(p[r_j])_l = u(p)_l \ \forall l \neq k,\, k'\,,$$

and

$$u(p)_{k'} = u(p[r_j])_{k'} = u(q[r_j])_{k'} \in \pi_{k'}(u(q[\Sigma_j]))\,.$$

On the other hand, since $u(q[\Sigma_j])$ is not an atom, there is $s_j \in \Sigma_j$ such that

$$u(q[s_j])_{k'} \neq u(p)_{k'}\,.$$

Note that $u(p)_{k'} = u(p[s_j])_{k'}$. Therefore, since $q[s_j]$ and $p[s_j]$ differ only by on component (namely j'), by part (1) we have

$$u(q[s_j])_l = u(p[s_j])_l \;\forall l \neq k'\,.$$

As a consequence,

$$u(q)_k = u(q[s_j])_k = u(p[s_j])_k \in \pi_k(u(p[\Sigma_j]))\,.$$

To summarize, we have proved that $u(p)_l = u(q)_l$ for all $l \neq k$, k', $u(p)_{k'} \in \pi_{k'}(u(q[\Sigma_j]))$, and $u(q)_k \in \pi_k(u(p[\Sigma_j]))$. As a consequence, the statement follows from Lemma 6.4 part (2).

(2.2) Since $\mathcal{L}_{j'}$ is connected, there is $s_{j'} \in p_{j'} \vee q_{j'}$ such that $p_{j'} \in q_{j'} \vee s_{j'}$ and $q_{j'} \in p_{j'} \vee s_{j'}$. Let $r = p[s_{j'}]$. Then, by Lemma 2.7.3,

$$r[\Sigma_j] \subseteq p[\Sigma_j] \vee q[\Sigma_j]\,, \qquad p[\Sigma_j] \subseteq q[\Sigma_j] \vee r[\Sigma_j] \quad \text{and} \quad q[\Sigma_j] \subseteq p[\Sigma_j] \vee r[\Sigma_j]\,.$$

Now, by part (2.1),

$$u(p[\Sigma_j] \vee q[\Sigma_j]) = u(p[\Sigma_j]) \vee u(q[\Sigma_j]) = u(p[\Sigma_j]) \cup u(q[\Sigma_j])\,.$$

As a consequence, by part (1), $u(r[\Sigma_j]) \subseteq u(p[\Sigma_j])$ or $u(r[\Sigma_j]) \subseteq u(q[\Sigma_j])$. Assume for instance that $u(r[\Sigma_j]) \subseteq u(p[\Sigma_j])$. Then

$$u(q[\Sigma_j]) \subseteq u(p[\Sigma_j] \vee r[\Sigma_j]) = u(p[\Sigma_j]) \vee u(r[\Sigma_j]) = u(p[\Sigma_j])\,,$$

a contradiction. Hence, we have proved that if p, $q \in \mathbf{\Sigma}$ differ only by one component, then $f(j,p) = f(j,q)$.

(2.3) Suppose now that p and q differ by more than one component. Since Ω is finite, there is $s_1, \ldots, s_n \in \mathbf{\Sigma}$ such that $s^1 = q$, $s^n = p$, and such that for all $1 \leq i \leq n-1$, s^i and s^{i+1} differ only by one component. Therefore,

$$f(j,q) = f(j,s^1) = f(j,s^2) = \cdots = f(j,s^n) = f(j,p)\,,$$

and we are done.]

(3) Let $p_0 \in \mathbf{\Sigma}$. Define $f\colon \Omega \to \Omega$ as $f(i) := f(i,p_0)$. Note that by part (2), f does not depend on the choice of p_0. *Claim*: The map f is surjective. [Proof: Let $k \in \Omega$. Suppose that for all $i \in \Omega$ $f(i) \neq k$. Let p, $q \in \mathbf{\Sigma}$ that differ only by one component, say j. Then $p[\Sigma_j] = q[\Sigma_j]$, therefore $u(p[\Sigma_j]) = u(q[\Sigma_j])$. Moreover, by part (1), there is $k' \neq k$ such $u(p[\Sigma_j]) \subseteq u(p)[\Sigma_{k'}]$. As a consequence, $u(p)_k = u(q)_k$.

Since Ω is finite, by the same argument as in part (2.3), we find that $u(p)_k = u(q)_k$, for all p, $q \in \mathbf{\Sigma}$. As a consequence, $u(\mathbf{\Sigma}) \subseteq \pi_k^{-1}(u(p_0)_k)$, a contradiction, since u is large.]

(4) Let $p_0 \in \mathbf{\Sigma}$ and $j \in \Omega$. Define $v_j \colon \mathcal{L}_j \to \mathcal{L}_{f(j)}$ as

$$v_j(a_j) := \pi_{f(j)}(u(p_0[a_j, j]))\,.$$

Claim: v_j does not depend on the choice of p_0. [Proof: Let $q \in \mathbf{\Sigma}$ that differs from p_0 only by one component, say $j' \neq j$. Then, by Lemma 2.7.3, we have

$$\begin{aligned}\pi_{f(j)}(u(q[a_j])) &= \pi_{f(j)}(u(\bigvee\{q[r_j]\ ;\ r_j \in a_j\}))\\ &= \pi_{f(j)}(\bigvee\{u(q[r_j])\ ;\ r_j \in a_j\}) = \bigvee_{r_j \in a_j} u(q[r_j])_{f(j)}\,,\end{aligned}$$

and the same formula holds for $\pi_{f(j)}(u(p_0[a_j]))$. Now

$$u(q[r_j])_{f(j)} = \pi_{f(j)}(u(s[\Sigma_{j'}])) = u(p_0[r_j])_{f(j)}\,,$$

where $s = q[r_j]$. As a consequence, $\pi_{f(j)}(u(q[a_j])) = \pi_{f(j)}(u(p_0[a_j]))$.

Since Ω is finite, by the same argument as in part (2.3), we find that

$$\pi_{f(j)}(u(q[a_j])) = \pi_{f(j)}(u(p_0[a_j]))$$

for all $q \in \mathbf{\Sigma}$.]

It remains to check that v_j preserves arbitrary joins. Let $\omega \subseteq \mathcal{L}_j$. Then, by Lemma 2.7.3,

$$\begin{aligned}v_j(\bigvee \omega) &= \pi_{f(j)}(u(p_0[\bigvee \omega])) = \pi_{f(j)}(u(\bigvee\{p_0[x, j]\ ;\ x \in \omega\}))\\ &= \pi_{f(j)}(\bigvee\{u(p_0[x, j])\ ;\ x \in \omega\})\\ &= \bigvee_{x\in\omega} \pi_{f(j)}(u(p_0[x, j])) = \bigvee_{x\in\omega} v_j(x)\,.\end{aligned}$$

$\square$

Corollary 6.6. *If the u in Theorem 6.5 is an automorphism, then all v_i's are isomorphisms.*

Theorem 6.7. *If the u in Theorem 6.5 is an automorphism, the statement remains true if for all $i \in \Omega$, $\mathcal{L}_i$ is weakly connected.*

Proof. The proof is similar as in Theorem 6.5. We only sketch the arguments that must be modified. Note that since u is an automorphism and $\mathcal{L}_i \neq 2$ for all $i \in \Omega$, u is large.

(1) Since $\mathcal{L}_j$ is weakly connected, there is $\gamma_0 \in \sigma^j$ such that $p_j \in A_j^{\gamma_0}$. By hypothesis and Lemma 2.7.3, for all $q_j \in A_j^{\gamma_0}$, $p \vee (p[q_j])$ contains a third atom, hence also $u(p) \vee u(p[q_j])$ since u is injective. As a consequence, there is $k_{\gamma_0} \in \Omega$ such that $u(p[A_j^{\gamma_0}]) \subseteq u(p)[\Sigma_{k_{\gamma_0}}]$. Moreover, since u is injective, by the third hypothesis in Definition 6.1, the map $\gamma \mapsto k_\gamma$ is constant. Therefore, since $\bigcup A_j^\gamma = \Sigma_j$, we find that $u(p[\Sigma_j]) \subseteq u(p)[\Sigma_k]$.

(2) Take p, $q \in \mathbf{\Sigma}$ that differ only by one component such that $q_{j'}$ and $p_{j'}$ are in the same A_j^{γ}.

(2.2) By hypothesis, $p_{j'} \vee q_{j'}$ contains a third atom, say $r_{j'}$, therefore $r[\Sigma_j] \subseteq p[\Sigma_j] \vee q[\Sigma_j]$, hence, $u(r[\Sigma_j]) = u(p[\Sigma_j])$ or $u(r[\Sigma_j]) = u(q[\Sigma_j])$, a contradiction since u is injective. As a consequence, $f(j,q) = f(j,p)$. Now, since u is injective, by the third hypothesis in definition 6.1, we find that $f(j,q) = f(j,p)$, for all p, $q \in \mathbf{\Sigma}$ that differ only by the component j'. □

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